Algebraic geometry 2 Exam Summer Term 2026

Solutions

Problem 1

1) True

2) False

3) True

4) False. $k[X,Y]/X^2$ is not an integral domain.

5) False. If A is not Noetherian, then $\mathbb{P}^n_A$ is not Noetherian.

6) True

7) False. Example: $X = \operatorname{Spec} \mathbb{C}$, $S = \operatorname{Spec} \mathbb{R}$

Both projections $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$

a just projection to one point.

So $\forall\, \delta \in \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$, we have $p_1(\delta) = p_2(\delta)$ but $\Delta \neq \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$

8) False, $Z \cap T =$ ~~Spec A/...~~ $\operatorname{Spec} A/_{I+J}$

Remark In terms of Alg. geom 1, we have

$$I(Z \cap T) = \sqrt{I(Z) + I(T)}.$$

9) True

10) True

Problem 2

1) $x \notin (x^2, xy) \Rightarrow \bar{x} \neq 0$

$$A/(\bar{x}) \simeq k[x,y]/(x^2, xy, x) \simeq k[x,y]/(x) \simeq \underset{\substack{\uparrow \\ \text{integral domain}}}{k[y]}$$

Hence, $(\bar{x})$ is prime in A $\rightsquigarrow$ point $\eta \in \operatorname{Spec} A$.

2) Let $\rho \in \operatorname{Spec} A$. We have $0 = \bar{x}^2 \in \rho \underset{\rho \text{ prime}}{\Longrightarrow} \bar{x} \in \rho$

$\Rightarrow$ $\cancel{\rho \in V(\eta)}$ $\rho \in V((\bar{x})) \Rightarrow V((\bar{x})) = X$

Recall that for a prime ideal $q \in \operatorname{Spec} A$ we have $\overline{\{q\}} = V(q)$

Hence, $\overline{\{\eta\}} = V((\bar{x})) = X$.

3) Clearly, $\bar{y} \notin (\bar{x})$. Hence $\eta \in D(\bar{y})$ and by (2) $\overline{D(\bar{y})} = X$

4)
$$\begin{array}{ccccc} \bar{x} & \in & A & = & \mathcal{O}_X(X) \\ \downarrow & & & & \downarrow \\ \frac{\bar{x}}{1} & \in & A_{\bar{y}} & = & \mathcal{O}_X(U) \\ & & & & \| \\ & & & & D(\bar{y}) \end{array}$$

Since $\bar{x}\bar{y} = 0$, by def. of the localization $A_{\bar{y}}$, we get

$\frac{\bar{x}}{1} = 0$ in $\cancel{\mathcal{O}_X(U)}$ $A_{\bar{y}} = \mathcal{O}_X(U)$

Problem 3

1) Let $\mathfrak{p} \in \mathcal{U}$.

Since $a|_{\mathcal{U}} = 0$ the stalk corresponding to the section $a \in \mathcal{O}_X(X)$ at point $\mathfrak{p}$ is zero

$$\begin{array}{ccc} \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_{X,\mathfrak{p}} \\ \cong & & \cong \\ A & \longrightarrow & A_{\mathfrak{p}} \\ a & \longmapsto & \frac{a}{1} = 0 \text{ in } A_{\mathfrak{p}} \end{array}$$

Hence, $sa = 0$ for some $s \notin \mathfrak{p}$ $\underset{\mathfrak{p}\text{ prime}}{\Rightarrow}$ $a \in \mathfrak{p}$ $\Rightarrow$ $\mathfrak{p} \in V(a)$

It follows, that $\mathcal{U} \subset \underset{\text{closed in } X}{\underset{\uparrow}{V(a)}}$

Since $\mathcal{U}$ is dense and $V(a)$ is close, we conclude: $V(a) = X$

Thus, a is nilpotent in A.

2) X reduced $\Rightarrow$ $\mathcal{O}_X(X) = A$ has no non-trivial nilpotent elements.

Now the statement follows from (1).

3) Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

and $\varphi, \psi : B \to A$ correspond to $f, g : X \to Y$ respectively.

Since two compositions ~~$\mathcal{U} \subset \operatorname{Spec} A$~~ $\mathcal{U} \overset{}{\underset{i}{\subseteq}} \operatorname{Spec} A \overset{f}{\underset{g}{\rightrightarrows}} \operatorname{Spec} B$

are equal, the morphism of sheaves $(f \circ i)^\# = (g \circ i)^\#$

Taking global sections, we get that the following compositions are equal: $\rho \circ \varphi = \rho \circ \psi$

$$\begin{array}{ccccc} \mathcal{O}_X(\mathcal{U}) & \xleftarrow[\rho]{} & \mathcal{O}_X(X) & \overset{\varphi}{\underset{\psi}{\leftleftarrows}} & \mathcal{O}_Y(Y) \\ & \nearrow & \cong & & \cong \\ & \text{restriction} & A & & B \end{array}$$

~~So~~ Note that ρ is injective by (2). Hence, $\rho \circ \varphi = \rho \circ \psi \Rightarrow \varphi = \psi \Rightarrow f = g$.

4) First let us prove the statements from the hint:

(i) By universal property of $Y\times_S Y$ we have

$$(f,g)|_U = (f|_U, g|_U) = (f|_U, f|_U) = (f,f)|_U.$$

(ii) Consider the morphism $\Delta\circ f$

$$\begin{tikzcd}
& X \arrow[d, "f"] \arrow[ddl, dashed, "f"'] & \\
& Y \arrow[d, "\Delta"] \arrow[dl, dashed, "id"'] & \\
Y \arrow[dr] & Y\times_S Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

Since $p_1\circ(\Delta\circ f) = (p_1\circ\Delta)\circ f \underset{\substack{\Delta \text{ is the} \\ \text{diagonal} \\ \text{morphism} \\ \Delta=(id,id)}}{=\!=\!=} (id)\circ f = f$

and similarly $p_2\circ(\Delta\circ f) = f$, we have by the UP of $Y\times_S Y$, that $(f,f) = \Delta\circ f$

By (i) we have $(f,g)(U) = (f,f)(U) = \Delta(f(U))$

Hence, $(f,g)(U) \subseteq \underbrace{\operatorname{Im}\Delta}_{\substack{\text{closed,} \\ \text{since } Y \text{ is separated over } S}} \subset Y\times_S Y$

We get $U \subset \underbrace{(f,g)^{-1}(\operatorname{Im}\Delta)}_{\text{closed in } X} \underset{U \text{ is dense}}{\Longrightarrow} (f,g)^{-1}(\operatorname{Im}\Delta) = X$

$\Rightarrow \operatorname{Im}(f,g) \subseteq \operatorname{Im}\Delta$

Let $x\in X$ and ~~set~~ $\delta = (f,g)(x)$

Then $p_1(\delta) = f(x)$ and $p_2(\delta) = g(x)$. But since $\delta\in\operatorname{Im}\Delta$, $p_1(\delta) = p_2(\delta)$ and $f(x) = g(x)$

5) We can cover $Y = \bigcup_{i \in I} V_i = \operatorname{Spec} B_i$ by affine open subsets $V_i = \operatorname{Spec} B_i$

~~It is enough to show~~

Note that $f^{-1}(V_i) = g^{-1}(V_i)$ by (4).

Set $\mathcal{U}_i := f^{-1}(V_i) = g^{-1}(V_i)$

It is enough to show that $f_{|\mathcal{U}_i} = g_{|\mathcal{U}_i} : \mathcal{U}_i \longrightarrow V_i$

$\forall i \in I$

Now we cover $\mathcal{U}_i$ by affine opens $\mathcal{U}_i = \bigcup_{j \in J} U_{ij} = \operatorname{Spec} A_{ij}$

and it is enough to show:

$$f|_{\mathcal{U}_{ij}} = g|_{\mathcal{U}_{ij}} : \underset{\operatorname{Spec} A_{ij}}{\underset{\shortparallel}{\mathcal{U}_{ij}}} \longrightarrow \underset{\operatorname{Spec} B_i}{\underset{\shortparallel}{V_i}}$$

But since $\mathcal{U}$ is dense in X, $\mathcal{U} \cap \mathcal{U}_{ij}$ is open and dense in $\mathcal{U}_{ij}$.

Hence, $f|_{\mathcal{U}_{ij} \cap \mathcal{U}} = g|_{\mathcal{U}_{ij} \cap \mathcal{U}}$

dense in $\mathcal{U}_{ij}$

By question (3) (since $\mathcal{U}_{ij}$ is reduced $\Leftarrow$ X is reduced)

we get $f|_{\mathcal{U}_{ij}} = g|_{\mathcal{U}_{ij}}$

Problem 4 (5)

1) Let $i_x : \operatorname{Spec} \mathcal{O}_{X,x} \to V \subset X$ for another affine open $V = \operatorname{Spec} B$, $x \in V$

Then $x \in \mathcal{U} \cap V$ and ~~we have~~ $\exists$ an affine open $x \in W = \operatorname{Spec} C \subset \mathcal{U} \cap V$ (affine opens subsets form a basis of topology on X)

Note that the natural ring hom. $\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_{X,x} = \varinjlim_{x \in \mathcal{U}} \mathcal{O}_X(\mathcal{U})$ factors through $\mathcal{O}_X(W)$ $(x \in W \subset \mathcal{U})$

We get the commutative diagramm.

$$\begin{tikzcd}
A = \mathcal{O}_X(\mathcal{U}) \arrow[dr] \arrow[drr] & & \\
& C = \mathcal{O}_X(W) \arrow[r] & \mathcal{O}_{X,x} \\
B = \mathcal{O}_X(V) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

By taking Spec we get the comm. diag.

$$\begin{tikzcd}
& & \mathcal{U} = \operatorname{Spec} A \\
\operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\mu"'] \arrow[urr, "j_x"] \arrow[drr, "i_x"'] & W = \operatorname{Spec} C \arrow[ur, hook] \arrow[dr, hook] & \\
& & V = \operatorname{Spec} B
\end{tikzcd}$$

open embeddings.

Since both j_x and i_x factors through the common open W, we get $j_x = i_x$.
and j_x does not depend on the choice of $\mathcal{U}$.

2) Moreover, ~~in the same way~~ we see that for any open neighborhood $\widetilde{W}$ of x, $\operatorname{Im} j_x \subset \widetilde{W}$ (same argument as in 1)

Hence, $\mathrm{Im}\, j_x \subseteq \bigcap_{x \in \mathcal{U}} \mathcal{U}$

Let now $y \in \bigcap_{x \in \mathcal{U}} \mathcal{U}$ (~~that is $x \in \overline{\{y\}}$ by the hint~~)

let $\mathfrak{p} = x \in \mathcal{U} = \mathrm{Spec}\, A$, then $\mathfrak{q} = y \in \mathcal{U}$ ($\mathfrak{p}, \mathfrak{q}$ prime ideals in A)

then

$$j_x : \mathrm{Spec}\, \mathcal{O}_{X,x} \longrightarrow \mathrm{Spec}\, A$$

~~Canonical~~ Natural ring hom. $\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_{X,x}$

corresponds to the canonical hom $A \longrightarrow A_{\mathfrak{p}}$

So we have $j_x : \mathrm{Spec}\, A_{\mathfrak{p}} \longrightarrow \mathrm{Spec}\, A \ (\subset X)$

$A_{\mathfrak{p}}$ is a localization of A

$$\mathrm{Im}\, j_x = \{ \tilde{\mathfrak{p}} \in \mathrm{Spec}\, A \mid \tilde{\mathfrak{p}} \subset \mathfrak{p} \}.$$

Now $y = \mathfrak{q} \in$ every open in $\mathrm{Spec}\, A$ which contain $x = \mathfrak{p}$

$\underset{\text{hint}}{\Longleftrightarrow}$ $\mathfrak{p} = x \in \overline{\{y\}} = \cancel{V(\mathfrak{q})}$ $\overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$ $\Longleftrightarrow$ $\mathfrak{q} \subset \mathfrak{p}$

$\Longleftrightarrow$ $\mathfrak{q} \in \mathrm{Im}\, j_x$

Problem 5

1) Assume w.l.o.g $\alpha_0 \neq 0$ and hence ~~$\alpha_0 = 1$~~

we can assume $\alpha_0 = 1$.

Then

$$m_\alpha = \langle X_j - \alpha_j X_0 \rangle_{j=1,\dots,n}.$$

~~Consider F~~ $m_\alpha \in V_+(F) \iff F \in m_\alpha$

$\iff$ class of $F = 0$ in $k[X_0, \dots, X_n]/m_\alpha \simeq k[X_0]$

here we have $\overline{X}_j = \alpha_j \overline{X}_0$

class of $F(X_0, X_1, \dots, X_n) = F(\overline{X}_0, \overline{X}_1, \dots, \overline{X}_n)$

$= F(\overline{X}_0, \alpha_1 \overline{X}_0, \dots, \alpha_n \overline{X}_n) = \cancel{X_0}\ F(1, \alpha_1, \dots, \alpha_n)\ \overline{X}_0^{\deg F}$

Hence class of $F = 0 \iff F(\alpha_0, \dots, \alpha_n) = 0$

2) $m_\alpha \in D_+(F) \iff m_\alpha \notin V_+(F) \underset{(1)}{\iff} F(\alpha_0, \dots, \alpha_n) \neq 0$.

3) From the lecture we know that $D_+(F)$ is affine $\underset{\text{Spec } k[X_0,\dots,X_n]_{(F)}}{\cong}$.

Let $y_1, \dots, y_m$ correspond to points $\alpha_1, \dots, \alpha_m \in \mathbb{P}^n(k^n)$

$$\alpha_i = [\alpha_{i0} : \alpha_{i1} : \dots : \alpha_{in}]$$

By (2) it is enough to find a hom. polynomial F,

s.t. $F(\alpha_1) = \dots = F(\alpha_m) \neq 0$

Take $F = b_0 X_0 + \dots + b_n X_n$ of degree 1

We can find coeff. $b_0, \dots, b_n$ s.t. $F(\alpha_1) = \dots = F(\alpha_m) \neq 0$

Otherwise non-zero polynomial on $b_0, \dots, b_n$ $F(\alpha_1) \cdot \dots \cdot F(\alpha_m)$ is zero $\forall\, b_0, \dots, b_n \in k$

~~not~~ not-possible, if k is alg. closed.