

Solutions

Problem 1

1) True

2) False

3) True

4) False, $k[X, Y]/X^2$ is not an integral domain.

5) False. If A is not Noetherian, then R_A^n is not Noetherian.

6) True

7) False. Example: $X = \text{Spec } \mathbb{C}$, $S = \text{Spec } \mathbb{R}$

Both projections $\text{Spec } \mathbb{C} \times_{\mathbb{R}} \text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{C}$

a just projection to one point.

So $\forall \delta \in \text{Spec } \mathbb{C} \times_{\mathbb{R}} \text{Spec } \mathbb{C}$, we have

$$p_1(\delta) = p_2(\delta) \text{ but } \Delta \neq \text{Spec } \mathbb{C} \times_{\mathbb{R}} \text{Spec } \mathbb{C}$$

8) False, $Z \cap T = \text{Spec } A / \sqrt{I+J} \text{Spec } A / I+J$

Remark In terms of Alg. geom 1, we have

$$I(Z \cap T) = \sqrt{I(Z) + I(T)}.$$

9) True

10) True

Problem 2

(2)

$$1) \quad x \notin (x^2, xy) \Rightarrow \bar{x} \neq 0$$

$$A/(\bar{x}) \simeq k[x, y]/(x^2, xy, x) \simeq k[x, y]/(x) \simeq k[y]$$

$\uparrow$
integral domain

Hence, $(\bar{x})$ is prime $\rightsquigarrow$ point $\eta \in \text{Spec } A$.

$$2) \quad \text{Let } \mathfrak{p} \in \text{Spec } A. \quad \text{We have } 0 = \bar{x}^2 \in \mathfrak{p} \xRightarrow{\mathfrak{p} \text{ prime}} \bar{x} \in \mathfrak{p}$$

$$\Rightarrow \mathfrak{p} \in V(\eta) \quad \mathfrak{p} \in V((\bar{x})) \Rightarrow V((\bar{x})) = X$$

Recall that for a prime ideal $\mathfrak{q} \in \text{Spec } A$
we have $\overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$

$$\text{Hence, } \overline{\{\eta\}} = V((\bar{x})) = X.$$

$$3) \quad \text{Clearly, } \bar{y} \notin (\bar{x}). \quad \text{Hence } \eta \in D(\bar{y}) \quad \text{and by (2)} \quad \overline{D(\bar{y})} = X$$

$$4) \quad \begin{array}{ccc} \bar{x} \in A & = & \mathcal{O}_x(X) \\ \downarrow & & \downarrow \\ \frac{\bar{x}}{1} \in A_{\bar{y}} & = & \mathcal{O}_x(U) \\ & & \text{" } D(\bar{y}) \end{array}$$

Since $\bar{x}\bar{y} = 0$, by def. of the localization $A_{\bar{y}}$, we get

$$\frac{\bar{x}}{1} = 0 \quad \text{in } \mathcal{O}_x(U) \quad A_{\bar{y}} = \mathcal{O}_x(U)$$

1) Let $p \in U$.

Since $a|_U = 0$ the stalk corresponding to the section $a \in \mathcal{O}_X(X)$ at point p is zero

$$\begin{array}{ccc} \mathcal{G}_X(X) & \longrightarrow & \mathcal{G}_{X,p} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_p \\ a & \longmapsto & \frac{a}{1} = 0 \text{ in } A_p \end{array}$$

Hence, $sa = 0$ for some $s \notin \mathfrak{p} \Rightarrow a \in \mathfrak{p} \Rightarrow \mathfrak{p} \in V(a)$
 $\mathfrak{p}$ prime

It follows, that $U \subset V(a)$
 $\uparrow$
 closed in X

Since U is dense and $V(a)$ is closed, we conclude: $V(a) = X$
Thus, a is nilpotent in A .

2) X reduced $\Rightarrow \mathcal{O}_X(X) = A$ has no non-trivial nilpotent elements.
Now the statement follows from (1).

3) Let $X = \text{Spec } A$, $Y = \text{Spec } B$
and $\varphi, \psi: B \rightarrow A$ correspond to $f, g: X \rightarrow Y$ respectively.

Since two compositions $U \subset \text{Spec } A \xrightarrow{f} \text{Spec } B$ are equal, the morphism of sheaves $(f \circ i)^\# = (g \circ i)^\#$

Taking global sections, we get that the following compositions are equal: $\rho \circ \varphi = \rho \circ \psi$

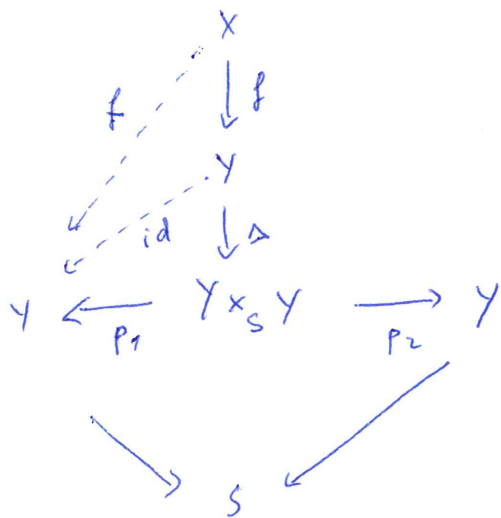
$$\mathcal{D}_X(\mathcal{U}) \xleftarrow[\rho]{\varphi} \mathcal{D}_X(\mathcal{X}) \xleftarrow[\psi]{\varphi} \mathcal{D}_Y(\mathcal{Y})$$

* Note that g is ^{restriction} injective by (2). Hence, $g \circ \varphi = g \circ \psi \Rightarrow \varphi = \psi \Rightarrow f = g$.

4) First let us prove the statements from the hint:
 (i) By universal property of $Y \times_S Y$ we have

$$(f, g)|_U = (f|_U, g|_U) = (f|_U, f|_U) = (f, f)|_U.$$

(ii) Consider the morphism $\Delta \circ f$



Since $p_1 \circ (\Delta \circ f) = (p_1 \circ \Delta) \circ f \stackrel{\Delta \text{ is the diagonal morphism } \Delta = (\text{id}, \text{id})}{=} (\text{id}) \circ f = f$

and similarly $p_2 \circ (\Delta \circ f) = f$, we have by the UP of $Y \times_S Y$, that $(f, f) = \Delta \circ f$

By (i) we have $(f, g)(u) = (f, f)(u) = \Delta(f(u))$

Hence, $(f, g)(u) \in \underbrace{\text{Im } \Delta}_{\text{closed}} \subset Y \times_S Y$

since Y is separated over S

We get $U \subset \underbrace{(f, g)^{-1}(\text{Im } \Delta)}_{\text{closed in } X} \implies (f, g)^{-1}(\text{Im } \Delta) = X$
 U is dense

$\implies \text{Im } (f, g) \subseteq \text{Im } \Delta$

Let $x \in X$ and $\delta \in \delta = (f, g)(x)$

Then $p_1(\delta) = f(x)$ and $p_2(\delta) = g(x)$. But since $\delta \in \text{Im } \Delta$, $p_1(\delta) = p_2(\delta)$ and $f(x) = g(x)$

5) We can cover $Y = \bigcup_{i \in I} V_i = \text{Spec } B_i$ by affine open subsets $V_i = \text{Spec } B_i$

~~It is enough to show~~

Note that $f^{-1}(V_i) = g^{-1}(V_i)$ by (4).

Set $U_i := f^{-1}(V_i) = g^{-1}(V_i)$

It is enough to show that $f|_{U_i} = g|_{U_i} : U_i \rightarrow V_i$

$\forall i \in I$

Now we cover U_i by affine opens $U_i = \bigcup_{j \in J} U_{ij} = \text{Spec } A_{ij}$

and it is enough to show:

$$f|_{U_{ij}} = g|_{U_{ij}} : \underbrace{U_{ij}}_{\text{Spec } A_{ij}} \longrightarrow \underbrace{V_i}_{\text{Spec } B_i}$$

But since U is dense in X , $U \cap U_{ij}$ is open and dense in U_{ij} .

Hence, $f|_{U_{ij} \cap U} = g|_{U_{ij} \cap U}$

$\nwarrow$
 dense in U_{ij}

By question (3) (since U_{ij} is reduced $\Leftarrow X$ is reduced)

we get $f|_{U_{ij}} = g|_{U_{ij}}$

Problem 4

(5)

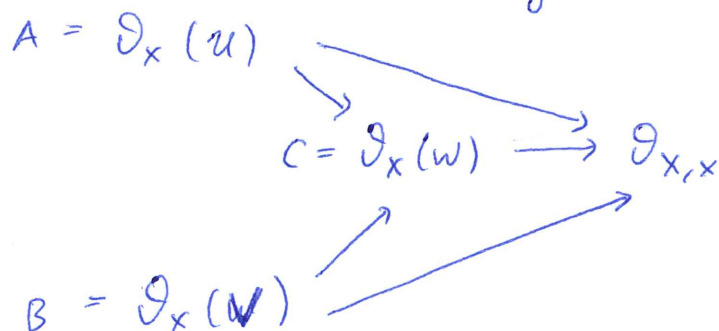
1) Let $i_x: \text{Spec } \mathcal{O}_{X,x} \rightarrow V \subset X$ for another affine open $V = \text{Spec } B, x \in V$

Then $x \in U \cap V$ and we have $\exists$ an affine open $x \in W = \text{Spec } C \subset U \cap V$ (affine opens subsets form a basis of topology on X)

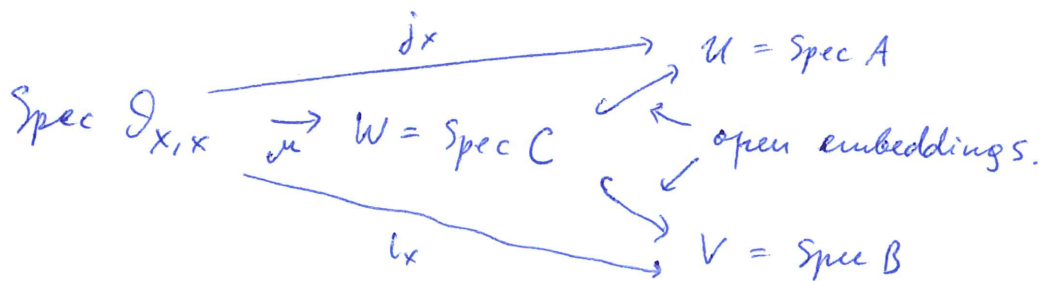
Note that the natural ring hom. $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$

factors through $\mathcal{O}_X(W)$ ($x \in W \subset U$)

We get the commutative diagram.



By taking Spec we get the comm. diag.



Since both j_x and i_x factors through the

common open W , we get $j_x = i_x$.

2) and j_x does not depend on the choice of U .

Moreover, in the same way we see that for any open neighborhood $\tilde{W}$ of x , $\text{Im } j_x \subset \tilde{W}$ (same argument as in 1)

Hence, $\text{Im } j_x \subseteq \bigcap_{x \in U} U$

Let now $y \in \bigcap_{x \in U} U$ (~~that is~~ $x \in \overline{\{y\}}$ by the hint)

let $\mathfrak{q} = x \in U = \text{Spec } A$, then $\mathfrak{q} = y \in U$ ($\mathfrak{p}, \mathfrak{q}$ prime ideals in A)

then $j_x: \text{Spec } \mathcal{O}_{X,x} \rightarrow \text{Spec } A$

Canonical Natural ring hom. $\mathcal{O}_x(U) \rightarrow \mathcal{O}_{x,x}$

corresponds to $A \rightarrow A_{\mathfrak{p}}$
the canonical hom

So we have $j_x: \text{Spec } A_{\mathfrak{p}} \rightarrow \text{Spec } A$ ($x = \mathfrak{p}$)

$A_{\mathfrak{p}}$ is a localization of A

$$\text{Im } j_x = \{ \tilde{\mathfrak{p}} \in \text{Spec } A \mid \tilde{\mathfrak{p}} \subset \mathfrak{p} \}.$$

Now $y = \mathfrak{q} \in$ every open in $\text{Spec } A$ which contains $x = \mathfrak{p}$

$$\stackrel{\text{hint}}{\Leftrightarrow} \mathfrak{p} = x \in \overline{\{y\}} = V(\mathfrak{q}) \quad \overline{\{y\}} = V(\mathfrak{q}) \quad \Leftrightarrow \mathfrak{q} \subset \mathfrak{p}$$

$$\Leftrightarrow \mathfrak{q} \in \text{Im } j_x$$

Problem 5

7

1) Assume w.l.o.g. $d_0 \neq 0$ and hence ~~not~~
we can assume $d_0 = 1$.

Then

$$m_d = \langle X_j - d_j X_0 \rangle_{j=1, \dots, n}.$$

Consider $m_d \in V_+(F) \Leftrightarrow F \in m_d$

$\Leftrightarrow$ class of $F = 0$ in $k[X_0, \dots, X_n] / m_d \simeq k[X_0]$

here we have $\bar{X}_j = d_j \bar{X}_0$

$$\text{class of } F(X_0, X_1, \dots, X_n) = F(\bar{X}_0, \bar{X}_1, \dots, \bar{X}_n)$$

$$= F(\bar{X}_0, d_1 \bar{X}_0, \dots, d_n \bar{X}_0) = \bar{X}_0^{\deg F} F(1, d_1, \dots, d_n)$$

Hence class of $F = 0 \Leftrightarrow F(d_0, \dots, d_n) = 0$

2) $m_d \in D_+(F) \Leftrightarrow m_d \notin V_+(F) \Leftrightarrow F(d_0, \dots, d_n) \neq 0$.
(1)

3) From the lecture we know that $D_+(F)$ is affine
 $\text{Spec } k[X_0, \dots, X_n]_{(F)}$.

Let $y_1, \dots, y_m$ correspond to points $d_1, \dots, d_m \in P^n(k^n)$

$$d_i = [d_{i0} : d_{i1} : \dots : d_{in}]$$

By (2) it is enough to find a hom. polynomial F ,
s.t. $F(d_1) = \dots = F(d_m) \neq 0$

Take $F = b_0 X_0 + \dots + b_n X_n$ of degree 1

We can find coeff. $b_0, \dots, b_n$ s.t. $F(d_1) = \dots = F(d_m) \neq 0$

Otherwise non-zero polynomial $F(d_1) \dots F(d_m)$ is zero $\forall b_0, \dots, b_n \in k$
on $b_0, \dots, b_n$ not-possible, if k is alg. closed.